Coherent States For $SU(3)$

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Abstract

We define coherent states for $SU(3)$ using six bosonic creation and annihilation operators. These coherent states are explicitly characterized by six complex numbers with constraints. For the completely symmetric representations $(n, 0)$ and $(0, m)$, only three of the bosonic operators are required. For mixed representations (n, m), all six operators are required. The coherent states provide a resolution of identity, satisfy the continuity property, and possess a variety of group theoretic properties. We introduce an explicit parameterization of the group $SU(3)$ and the corresponding integration measure. Finally, we discuss the path integral formalism for a problem in which the Hamiltonian is a function of $SU(3)$ operators at each site.

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1 Introduction

Coherent states have been used for a long time in different areas of physics [1, 2]. In condensed matter physics, coherent states for the Lie group $SU(2)$ have been extensively used to study Heisenberg spin systems using the path integral formalism [3, 4, 5, 6]. These studies have been generalized to systems with $SU(N)$ symmetry; these studies have usually been restricted to the completely symmetric representations [4, 7]. However, there is a recent discussion of coherent states for arbitrary irreducible representations of $SU(3)$ in Ref. [8]. The purpose of our work is to discuss a coherent state formalism which is valid for all representations of $SU(3)$, and to give an explicit characterization of them in terms of complex numbers and the states of some harmonic oscillators. (Our work differs in this respect from Ref. [8] which does not use harmonic oscillator operators to define the basis states). As we will see, this way of characterization is very similar to those used for the Heisenberg-Weyl and $SU(2)$ coherent states. But, there are also certain features (such as tracelessness) which are redundant in the simpler case of $SU(2)$.

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As additional motivation for our work, we should mention that there have been many other studies of $SU(3)$ in the recent mathematical physics literature, including the geometric phase for three-level systems [9] and the study of Clebsch-Gordon coefficients and the outer multiplicity problem [10]. These studies do not use coherent states; however our work is likely to shed new light on some of these studies. For instance, we will use two triplets of complex numbers z and w which are similar to the ones used in [10], except that we will normalize the triplets to unity. Similarly, it is well-known that the geometric phases in the different representations of $SU(2)$ may be obtained by integrating around a closed loop the overlap of two coherent states which differ infinitesimally from each other [5, 6]. In the same way, it should be possible to derive the geometric phases for $SU(3)$ representations from the coherent states discussed below.

The organization of the paper is as follows. Section 2 will motivate our ideas and techniques using two examples which are simpler than the $SU(3)$ group. We start with the standard group theoretical definitions of the coherent states of the Heisenberg-Weyl and $SU(2)$ groups. We then discuss another way of defining $SU(2)$ coherent states using the Schwinger or Holstein-Primakoff representation of the Lie algebra of $SU(2)$ [11] in terms of harmonic oscillator creation and annihilation operators. This definition is discussed in some detail as it can be extended to the $SU(3)$ group. We then establish its equivalence with the standard group theoretical coherent state definition [2]. In section 3, we generalize the $SU(2)$ Lie algebra in terms of harmonic oscillators to the $SU(3)$ group, and construct the irreducible representations of $SU(3)$. We describe the structure of $SU(3)$ matrices in an explicit way, and provide an integration measure for this 8-dimensional manifold. In section 4, we use this group structure to construct a set of $SU(3)$ coherent states which are explicitly characterized by a set of complex numbers which are equivalent to 8 real variables. We prove various identities expected for coherent states such as the the resolution of identity and a transformation from a particular coherent state to the general coherent state. In section 5, we provide an alternative set of coherent states for $SU(3)$ which require only 5 real variables; although these share some of the features of the coherent states defined in section 4, they have a few limitations arising from the smaller number of variables used. In section 6, we discuss how coherent states can be used to develop a path integral formalism for problems involving $SU(3)$ variables.

2 Heisenberg-Weyl and $SU(2)$ Coherent States

There are many definitions of coherent states used in the literature. However, the most essential ingredients common in all these definitions are the continuity and completeness properties [1].

1. These are states in a Hilbert space $\mathcal{H}$ associated which are characterized by a set of continuous variables $\{\vec{z}\}$, and the coherent states $|\vec{z}>$ are strongly continuous functions of the labels $\{\vec{z}\}$.

2. There exists a positive measure $d\mu(\vec{z})$ such that the unit operator $\mathcal{I}$ admits the resolution of identity

$$\mathcal{I} \;=\; \int \, d\mu(\vec{z}) \; |\vec{z}><\vec{z}| \;. \tag{1}$$

Given a group G, the coherent states in a given representation R are functions of q parameters denoted by $\{z_1, z_2, ...z_q\}$, and are defined as

$$|\vec{z}> \;\; \equiv \;\; T_R(g(\vec{z}))\; |0>_R \;\; . \tag{2}$$

Here $T_R(g(\vec{z}))$ is a group element in the representation R, and $|0>_R$ is a fixed vector belonging to R. In the simplest example of the Heisenberg-Weyl group, the Lie algebra contains three generators. It is defined in terms of creation annihilation operators $(a, a^\dagger)$ satisfying

$$[a, a^\dagger] = \mathcal{I}, \quad [a, \mathcal{I}] = 0, \quad [a^\dagger, \mathcal{I}] = 0 \; . \tag{3}$$

This algebra has only one infinite dimensional irreducible representation which can be characterized by occupation number states $|n> \equiv \frac{(a^\dagger)^n}{\sqrt{n!}}|0>$ with $n = 0, 1, 2....$ A generic group element in (2) can be characterized by $T(g) = \exp\;(i\alpha\mathcal{I} + za^\dagger - \bar{z}a)$ with an angle α and a complex parameter z. Therefore,

$$\begin{aligned} |\alpha, z>_\infty \;\; &= \;\; \exp(i\alpha)\; |z>, \\ |z> \;\; &= \;\; \exp(za^\dagger - \bar{z}a)\; |0> = \sum_{n=0}^{\infty} F_n(z)\; |n> \;\; , \end{aligned} \tag{4}$$

where the sum runs over all the basis vectors of the infinite dimensional representation, and

$$F_n(z) \;\; = \;\; \frac{z^n}{\sqrt{n!}} \; \exp(-|z|^2/2) \tag{5}$$

are the coherent state expansion coefficients. This feature, i.e., an expansion of the coherent states in terms of basis vectors of a given representation with analytic functions of complex variables $(F_n(z))$ as coefficients, will also be present in the case of $SU(2)$ and $SU(3)$ groups. It is easy to see that Eq. (4) provides a resolution of identity as in (1) with the measure $d\mu(z) = dzd\bar{z}$.

We now briefly review the next simplest example, i.e., the coherent states associated with the $SU(2)$ group. The $SU(2)$ Lie algebra is given by a set of three angular momentum operators $\{\vec{J}\} \equiv \{J_1, J_2, J_3\}$ or equivalently by $\{J_+, J_-, J_3\}$, $(J_\pm \equiv J_1 \pm i\; J_2)$ satisfying

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_3 \; . \tag{6}$$

The $SU(2)$ group has a Casimir operator given by $\vec{J} \cdot \vec{J}$, and the different irreducible representations are characterized by its eigenvalues $j(j+1)$, where j is an integer or half-odd-integer. A given basis vector in representation j is labeled by the eigenvalue m of J_3 as $|j, m>$. We characterize the SU(2) group elements U by the Euler angles, i.e, $U(\theta, \phi, \psi) \equiv exp - i\phi J_3 exp - i\theta J_2 exp - i\psi J_3$. The standard group theoretical definition (2) takes $|0>_j$ in (2) to be the highest weight state $|j, j>$ and is of the form:

$$\begin{aligned} |\hat{n}(\theta, \phi)>_j \;\; &= \;\; U(\theta, \phi, \psi)\; |j, j> \;\; , \\ &= \;\; \sum_{m=-j}^{+j} C_m(\theta, \phi)\; |j, m> \;\; , \end{aligned} \tag{7}$$

In (7), the coefficients $C_m(\theta,\phi)$ are given by,

$$C_m(\theta,\phi) = e^{-im\phi}[\frac{2j!}{(j+m)!(j-m)!}]^{\frac{1}{2}}[sin\frac{\theta}{2}]^{j-m}[cos\frac{\theta}{2}]^{j+m} \tag{8}$$

where we have ignored possible phase factors.

The algebra in Eq. (6) can also be realized in terms of a doublet of harmonic oscillator creation and annihilation operators $\vec{a} \equiv (a_1, a_2)$ and $\vec{a}^\dagger \equiv (a_1^\dagger, a_2^\dagger)$ respectively [11]. They satisfy the simpler bosonic commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$ with $i,j = 1,2$. The vacuum state is $|0,0>$. In terms of these operators,

$$J^a \;\equiv\; \frac{1}{2}\, a_i^\dagger\; (\sigma^a)_{ij}\; a_j\ , \tag{9}$$

where σ^a denote the Pauli matrices. (We will generally use the convention that repeated indices are summed over). It is easy to check that the operators in (9) satisfy the $SU(2)$ Lie algebra with the Casimir $\vec{J}\cdot\vec{J} \equiv \frac{1}{4}\vec{a}^\dagger\cdot\vec{a}(\vec{a}^\dagger\cdot\vec{a}+2)$. Thus the representations of $SU(2)$ can be characterized by the eigenvalues of the occupation number operator; the spin value j is equal to $(N_1+N_2)/2$ where N_1 and N_2 are the eigenvalues of $a_1^\dagger a_1$ and $a_2^\dagger a_2$ respectively.

With these harmonic oscillator creation and annihilation operators, another definition of $SU(2)$ coherent states is obtained by directly generalizing (4). We define a doublet of complex numbers (z_1, z_2) with the constraint $|z_1|^2 + |z_2|^2 = 1$; this gives 3 independent real parameters which define the sphere S^3. Let us parameterize

$$z_1 \;=\; \cos\chi\; e^{i\beta_1}\ , \quad \text{and} \quad z_2 \;=\; \sin\chi\; e^{i\beta_2}\ , \tag{10}$$

where $0 \le \chi \le \pi/2$ and $0 \le \beta_1, \beta_2 < 2\pi$. The the integration measure on this space takes the form

$$d\Omega_{S^3} \;=\; \frac{1}{\pi^2}\, dz_1\, d\bar{z}_1\, dz_2\, d\bar{z}_2\; \delta(|z_1|^2 + |z_2|^2 - 1) \;=\; \frac{1}{2\pi^2}\; \cos\chi\;\; \sin\chi\; d\chi\; d\beta_1\; d\beta_2\ , \tag{11}$$

where we have introduced a normalization factor so that $\int d\Omega_{S^3} = 1$. The $SU(2)$ coherent state in the representation N is now defined as

$$\begin{aligned} |z_1, z_2>_{N=2j} \;&=\; \delta_{\vec{a}^\dagger\cdot\vec{a},N}\; \sqrt{N!}\; \exp\left(\vec{z}\cdot\vec{a}^\dagger\right)\; |0,0> \\ &=\; \sum_{N_1,N_2}{}'\; F_{N_1,N_2}\; |N_1, N_2>_j\ \ . \end{aligned} \tag{12}$$

In the second equation above, the $\sum'$ implies that only the terms satisfying the constraint $a^\dagger \cdot a = N \equiv 2j$ are included or equivalently that

$$N_1 + N_2 = N\ . \tag{13}$$

With (13), the states $|N_1, N_2>_j$ form a $(2j+1)$-dimensional representation of $SU(2)$. The expansion coefficients $F_{N_1,N_2}(z_1, z_2)$ are analytic functions of (z_1, z_2) and are given by

$$F_{N_1,N_2} \;\equiv\; \Big(\frac{N!}{N_1!N_2!}\Big)^{1/2}\; z_1^{N_1} z_2^{N_2}\ . \tag{14}$$

Eqs. (12) and (14) are similar to (4) and (5) respectively. This will be generalized to the $SU(3)$ case in section 3. It is easy to check that (12) provides the resolution of identity with the measure given in (11), namely,

$$\int d\Omega_{SU(2)} |z_1, z_2 >< z_1, z_2| \; = \; \frac{1}{N+1} \sum_{m=-j}^{j} |j, m >< j, m| \; . \tag{15}$$

Now we change variables from N_1 and $N_2 = 2j - N_1$ to $m = \frac{1}{2}(N_1 - N_2)$, and define

$$\omega \; \equiv \; \frac{z_1}{z_2} \; = \; e^{i\phi} \cot \frac{\theta}{2} \; . \tag{16}$$

These parameters are related to the ones given in (10) as $\theta = 2\chi$ and $\phi = \beta_1 - \beta_2$. We now consider an unit sphere S^2 with its south pole touching the point $\omega = 0$. The sphere is characterized by (θ, ϕ) where θ and ϕ are the polar and azimuthal angles respectively. Using the stereographic projection, it is easy to verify that

$$\begin{aligned} |z_1, z_2 >_j \quad &= \quad (z_1)^{2j} \sum_{m=-j}^{j} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} (\omega)^{(m-j)} |j, m > \\ &= \quad \Big(z_1 cos(\frac{\theta}{2}) \Big)^{2j} |\hat{n}(\theta, \phi) >_j \;\; , \end{aligned} \tag{17}$$

where we have again ignored possible phase factors. Eq. (17) can also be written as

$$|z_1, z_2 >_j \;\; = \;\; (z_1)^{2j} \exp(\frac{z_2}{z_1} J_-) |z_1 = 1, z_2 = 0 > \;\; , \tag{18}$$

where $|z_1 = 1, z_2 = 0 >_{N=2j} = |j, j >$ and we have used the fact that $J_- = a_2^\dagger a_1$. Eqs. (17) and (18) establish the equivalence between the group theoretical theoretical definition (7) and the one using Schwinger bosons (12).

The stationary subgroup of a particular coherent state is defined as the subgroup H of the full group G which leaves that coherent state invariant up to a phase; the coherent states are functions of the coset space G/H [2]. It is clear from the discussion above that the stationary subgroup of the $SU(2)$ coherent states is $U(1)$; therefore the coherent states correspond to the coset space $SU(2)/U(1) = S^2$ which is parameterized by the angles (θ, ϕ).

3 $SU(3)$ and its Representations

Let us first discuss a parameterization of $SU(3)$ matrices, i.e., 3×3 unitary matrices with unit determinant. To motivate this, let us first consider a parameterization of $SO(3)$ matrices. Consider a real vector of unit length of the form

$$\vec{p} \; = \; \begin{pmatrix} \sin\theta \; \cos\phi \\ \sin\theta \; \sin\phi \\ \cos\theta \end{pmatrix} \; . \tag{19}$$

The most general real vector q of unit length which is orthogonal to p is given by

$$\vec{q} \;=\; \begin{pmatrix} \cos\chi\;\cos\theta\;\cos\phi \;+\; \sin\chi\;\sin\phi \\ \cos\chi\;\cos\theta\;\sin\phi \;-\; \sin\chi\;\cos\phi \\ -\;\cos\chi\;\sin\theta \end{pmatrix} . \tag{20}$$

Finally, we define a third unit vector $\vec{r} = \vec{p}\times\vec{q}$, i.e., $r_1 = p_2 q_3 - p_3 q_2$ etc. Then a 3×3 matrix whose columns are given by the vectors p, q and r is an $SO(3)$ matrix.

We will now generalize the above construction to obtain an $SU(3)$ matrix. A complex vector of unit norm is given by

$$\vec{z} \;=\; \begin{pmatrix} \sin\theta\;\cos\phi\; e^{i\alpha_1} \\ \sin\theta\;\sin\phi\; e^{i\alpha_2} \\ \cos\theta\; e^{i\alpha_3} \end{pmatrix} , \tag{21}$$

where $0 \le \theta, \phi \le \pi/2$ and $0 \le \alpha_1, \alpha_2, \alpha_3 < 2\pi$. Then the integration measure for $\vec{z}$, which is equivalent to the sphere S^5, is given by

$$\begin{aligned} d\Omega_{S^5} \;&=\; \frac{2}{\pi^3}\; dz_1\; d\bar{z}_1\; dz_2\; d\bar{z}_2\; dz_3\; d\bar{z}_3\; \delta(|z_1|^2 + |z_2|^2 + |z_3|^2 - 1) \\ &=\; \frac{1}{\pi^3}\; \sin^3\theta\;\cos\theta\;\cos\phi\;\sin\phi\; d\theta\; d\phi\; d\alpha_1\; d\alpha_2\; d\alpha_3 \;, \end{aligned} \tag{22}$$

which has been normalized to make $\int d\Omega_{S^5} = 1$. The most general complex vector $\vec{w}$ of unit norm satisfying $\vec{z}\cdot\vec{w} = 0$ is given by

$$\vec{w} \;=\; \begin{pmatrix} e^{i(\beta_1-\alpha_1)}\;\cos\chi\;\cos\theta\;\cos\phi \;+\; e^{i(\beta_2-\alpha_1)}\;\sin\chi\;\sin\phi \\ e^{i(\beta_1-\alpha_2)}\;\cos\chi\;\cos\theta\;\sin\phi \;-\; e^{i(\beta_2-\alpha_2)}\;\sin\chi\;\cos\phi \\ -\;e^{i(\beta_1-\alpha_3)}\;\cos\chi\;\sin\theta \end{pmatrix} , \tag{23}$$

where $0 \le \chi \le \pi/2$ and $0 \le \beta_1, \beta_2 < 2\pi$ just as in the integration measure for S^3 in (11). We may now define a third complex vector of unit norm as $\vec{v} = \vec{\bar{z}} \times \vec{w}$, where $\vec{\bar{z}} \equiv \vec{z^\star}$. Then we can check that a 3×3 matrix whose columns are given by z, $\bar{w}$ and v, i.e.,

$$S \;=\; \begin{pmatrix} z_1 & \bar{w}_1 & \bar{z}_2 w_3 - \bar{z}_3 w_2 \\ z_2 & \bar{w}_2 & \bar{z}_3 w_1 - \bar{z}_1 w_3 \\ z_3 & \bar{w}_3 & \bar{z}_1 w_2 - \bar{z}_2 w_1 \end{pmatrix} \tag{24}$$

is an $SU(3)$ matrix.

The integration measure for the group $SU(3)$ is given by a product of (22) and (11) as [12]

$$d\Omega_{SU(3)} \;=\; \frac{1}{2\pi^5}\; \sin^3\theta\;\cos\theta\;\cos\phi\;\sin\phi\;\cos\chi\;\sin\chi\; d\theta\; d\phi\; d\chi\; d\alpha_1\; d\alpha_2\; d\alpha_3\; d\beta_1\; d\beta_2 \;, \tag{25}$$

which is normalized so that $\int d\Omega_{SU(3)} = 1$. To prove Eq. (25), we note that the matrix in (24) can be written as a product of two $SU(3)$ matrices, i.e., $S = A_3 A_2$, where

$$A_3 \;=\; \begin{pmatrix} \sin\theta\;\cos\phi\; e^{i\alpha_1} & \cos\theta\;\cos\phi\; e^{i\alpha_1} & -\sin\phi\; e^{-i\alpha_2 - i\alpha_3} \\ \sin\theta\;\sin\phi\; e^{i\alpha_2} & \cos\theta\;\sin\phi\; e^{i\alpha_2} & \cos\phi\; e^{-i\alpha_1 - i\alpha_2} \\ \cos\theta\; e^{i\alpha_3} & -\sin\theta\; e^{i\alpha_3} & 0 \end{pmatrix} , \tag{26}$$

and

$$A_2 \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\chi\; e^{-i\beta_1} & \sin\chi\; e^{i\beta_2 - i\alpha_1 - i\alpha_2 - i\alpha_3} \\ 0 & -\sin\chi\; e^{-i\beta_2 + i\alpha_1 + i\alpha_2 + 1\alpha_3} & \cos\chi\; e^{i\beta_1} \end{pmatrix} . \qquad (27)$$

The structure of the matrix A_3 is determined entirely by the three-dimensional complex vector which forms its first column; hence the integration measure corresponding to it is given by (22). The matrix A_2 is determined by the two-dimensional complex vector which forms its second column; its contribution to the integration measure is therefore given by (11). Note that although the parameter appearing in A_2 is $\beta_2 - \alpha_1 - \alpha_2 - \alpha_2$ instead of only β_2 as in (10), this makes no difference in the product measure given in (25) since the differentials $d\alpha_i$ already appear in the integration measure coming from A_3. Incidentally, this procedure generalizes to any $SU(N)$; the integration measure is given by a product of measures for S^{2N-1}, S^{2N-3}, ..., S^3 [12].

In short, we have defined two complex vectors $\vec{z} = (z_1, z_2, z_3)$ and $\vec{w} = (w_1, w_2, w_3)$ in (21) and (23). These satisfy the constraints

$$\begin{aligned} \bar{\vec{z}} \cdot \vec{z} \;&=\; |z_1|^2 + |z_2|^2 + |z_3|^2 \;\;=\;\; 1 \;, \\ \bar{\vec{w}} \cdot \vec{w} \;&=\; |w_1|^2 + |w_2|^2 + |w_3|^2 \;\;=\;\; 1 \;, \end{aligned} \qquad (28)$$

and

$$\vec{z} \cdot \vec{w} \;=\; z_1 w_1 + z_2 w_2 + z_3 w_3 \;=\; 0 \;. \qquad (29)$$

These constraints leave eight real degrees of freedom as required for $SU(3)$. We will take $\vec{z}$ and $\vec{w}$ to transform respectively as the 3 and $3^\star$ representation of $SU(3)$. Thus an $SU(3)$ transformation acts on the matrix S in Eq. (24) by multiplication from the left.

Let us now define two triplets of harmonic oscillator creation annihilation operators (a_i, b_i), i=1,2,3, satisfying

$$\begin{aligned} [a_i, a_j^\dagger] = \;\; & \delta_{ij} \;, \qquad && [b_i, b_j^\dagger] = \delta_{ij} \;, \\ [a_i, b_j] = \;\; & 0 \;, \qquad && [a_i, b_j^\dagger] = 0 \;. \end{aligned} \qquad (30)$$

We will often denote these two triplets by $(\vec{a}, \vec{b})$ and the two number operators by $N_a (\equiv \vec{a}^\dagger \cdot \vec{a})$ and $N_b (\equiv \vec{b}^\dagger \cdot \vec{b})$. Similarly, their vacuum state is denoted by $|\vec{0}_a, \vec{0}_b>$. Henceforth, we will ignore the subscripts a, b and will denote the vacuum state by $|\vec{0}, \vec{0}>$, and the eigenvalues of N_a, N_b by N and M respectively.

Now let λ^a, $a = 1, 2, ..., 8$ be the generators of $SU(3)$ in the fundamental representation; they satisfy the $SU(3)$ Lie algebra $[\lambda^a, \lambda^b] = i f^{abc} \lambda^c$. Let us define the following operators

$$Q^a = a^\dagger \lambda^a a - b^\dagger \lambda^{*a} b \;, \qquad (31)$$

where $a^\dagger \lambda^a a \equiv a_i^\dagger \lambda^a_{ij} a_j$, and $b^\dagger \lambda^{*a} b \equiv b_i^\dagger \lambda^{*a}_{ij} b_j$. To be explicit,

$$Q^3 \;\;=\;\; \frac{1}{2}\,(a_1^\dagger a_1 \;-\; a_2^\dagger a_2 \;-\; b_1^\dagger b_1 \;+\; b_2^\dagger b_2) \;,$$

$$
\begin{aligned}
Q^8 &= \frac{1}{2\sqrt{3}}\,(a_1^\dagger a_1 \;+\; a_2^\dagger a_2 \;-\; 2a_3^\dagger a_3 \;-\; b_1^\dagger b_1 \;-\; b_2^\dagger b_2 \;+\; 2b_3^\dagger b_3)\ ,\\
Q^1 &= \frac{1}{2}\,(a_1^\dagger a_2 \;+\; a_2^\dagger a_1 \;-\; b_1^\dagger b_2 \;-\; b_2^\dagger b_1)\ ,\\
Q^2 &= -\frac{i}{2}\,(a_1^\dagger a_2 \;-\; a_2^\dagger a_1 \;+\; b_1^\dagger b_2 \;-\; b_2^\dagger b_1)\ ,\\
Q^4 &= \frac{1}{2}\,(a_1^\dagger a_3 \;+\; a_3^\dagger a_1 \;-\; b_1^\dagger b_3 \;-\; b_3^\dagger b_1)\ ,\\
Q^5 &= -\frac{i}{2}\,(a_1^\dagger a_3 \;-\; a_3^\dagger a_1 \;+\; b_1^\dagger b_3 \;-\; b_3^\dagger b_1)\ ,\\
Q^6 &= \frac{1}{2}\,(a_2^\dagger a_3 \;+\; a_3^\dagger a_2 \;-\; b_2^\dagger b_3 \;-\; b_3^\dagger b_2)\ ,\\
Q^7 &= -\frac{i}{2}\,(a_2^\dagger a_3 \;-\; a_3^\dagger a_2 \;+\; b_2^\dagger b_3 \;-\; b_3^\dagger b_2)\ .
\end{aligned}
\tag{32}
$$

It can be checked that these operators satisfy the $SU(3)$ algebra amongst themselves, i.e, $[Q^a, Q^b] = if^{abc}Q^c$. Further,

$$
\begin{aligned}
[Q^a, a_i^\dagger] &= \lambda^a_{ji} a_j^\dagger\ , \qquad & [Q^a, b_i^\dagger] &= -\lambda^{*a}_{ji} b_j^\dagger\ ,\\
[Q^a, a^\dagger \cdot a] &= 0\ , \qquad & [Q^a, b^\dagger \cdot b] &= 0\ ,\\
[Q^a, a^\dagger \cdot b^\dagger] &= 0\ , \qquad & [Q^a, a \cdot b] &= 0\ .
\end{aligned}
\tag{33}
$$

From Eqs. (33), it is clear that the three states $a_i^\dagger|\vec{0},\vec{0}>$ with $(N=1, M=0)$ and $b_i^\dagger|\vec{0},\vec{0}>$ with $(N=0, M=1)$ transform respectively as the fundamental representation (3) and its conjugate representation ($3^\star$). By taking the direct product of N $\vec{a}^\dagger$'s and M $\vec{b}^\dagger$'s we can now form higher representations. We now define an operator

$$
O^{i_1 i_2 \ldots i_N}_{j_1 j_2 \ldots j_M} \equiv a^\dagger_{i_1} a^\dagger_{i_2} \ldots a^\dagger_{i_N} b^\dagger_{j_1} b^\dagger_{j_2} \ldots b^\dagger_{j_M}\ . \tag{34}
$$

Under $SU(3)$ transformation the states defined as $|\tilde{\psi}>_{(N,M)} \equiv O^{i_1 i_2 \ldots i_N}_{j_1 j_2 \ldots j_M}|\vec{0},\vec{0}>$ will all have $N_a = N$ and $N_b = M$, and will transform amongst themselves. Further, $|\tilde{\psi}> = N|\tilde{\psi}>$ and $N_b|\tilde{\psi}> = M|\tilde{\psi}>$. However, these do not form an irreducible representation because $\vec{a}\cdot\vec{b}$ and $\vec{a}^\dagger \cdot \vec{b}^\dagger$ are $SU(3)$ invariant operators (see (33)). A general basis vector in the irreducible representation (N, M) is obtained by subtracting the traces and completely symmetrizing in upper and lower indices [13]. More explicitly, a state in (N, M) representation is given by

$$
\begin{aligned}
|\psi>^{i_1,i_2,\ldots i_N}_{j_1,j_2,\ldots,j_M} \equiv \Big[O^{i_1 i_2 \ldots i_N}_{j_1 j_2 \ldots j_M} &+ L_1 \sum_{l_1=1}^{N} \sum_{k_1=1}^{M} \delta^{i_{l_1}}_{j_{k_1}} O^{i_1 i_2 .. i_{l_1-1} i_{l_1+1} .. i_N}_{j_1 j_2 .. j_{k_1-1} j_{k_1+1} \cdots j_M} \\
+ L_2 \sum_{l_1,l_2=1}^{N} \sum_{k_1,k_2=1}^{M} & \delta^{i_{l_1}}_{j_{k_1}} \delta^{i_{l_2}}_{j_{k_2}} O^{i_1 i_2 .. i_{l_1-1} i_{l_1+1} .. i_{l_2-1} i_{l_2+1} .. i_N}_{j_1 j_2 .. j_{k_1-1} j_{k_1+1} .. j_{k_2-1} j_{k_2+1} \cdots j_M} \\
+ L_3 \sum_{l_1,l_2,l_3=1}^{N} \sum_{k_1,k_2,k_3=1}^{M} & \delta^{i_{l_1}}_{j_{k_1}} \delta^{i_{l_2}}_{j_{k_2}} \delta^{i_{l_3}}_{j_{k_3}} O^{i_1 i_2 .. i_{l_1-1} i_{l_1+1} .. i_{l_2-1} i_{l_2+1} .. i_{l_3-1} i_{l_3+1} .. i_N}_{j_1 j_2 .. j_{k_1-1} j_{k_1+1} .. j_{k_2-1} j_{k_2+1} .. j_{k_3-1} j_{k_3+1} \cdots j_M} \quad + \ldots
\end{aligned}
$$

$$+L_Q \sum_{l_1,l_2,l_3,..,l_Q=1}^{N} \sum_{k_1,k_2,k_3,..,k_Q=1}^{M} \delta^{i_{l_1}}_{j_{k_1}} \delta^{i_{l_2}}_{j_{k_2}} .. \delta^{i_{l_Q}}_{j_{k_Q}} O^{i_1 i_2 .. i_{l_1-1} i_{l_1+1} .. i_{l_2-1} i_{l_2+1} .. i_{l_Q-1} i_{l_Q+1} .. i_N}_{j_1 j_2 .. j_{k_1-1} j_{k_1+1} .. j_{k_2-1} j_{k_2+1} .. j_{k_Q-1} j_{k_Q+1} ... j_M} \Big] |\vec{0},\vec{0}>, \tag{35}$$

where $Q = \mathrm{Min}(N, M)$,

$$L_q \equiv \frac{(-1)^q \ (a^\dagger \cdot b^\dagger)^q}{q!(N+M+1)(N+M)(N+M-1)...(N+M+2-q)} \ , \tag{36}$$

and all the sums in (35) are over different indices, i.e, $l_1 \neq l_2 ... \neq l_q$ and $k_1 \neq k_2 ... \neq k_q$. The coefficients in Eq. (36) are chosen to satisfy the tracelessness condition

$$\sum_{i_l, j_k=1}^{3} \delta^{i_l}_{j_k} |\psi>^{i_1,i_2,...i_N}_{j_1,j_2,...,j_M} = 0, \quad \text{for all} \quad l = 1,2...N, \quad \text{and} \quad k = 1,2...M \ . \tag{37}$$

For future purposes, a more compact notation for describing all the states given above is to write

$$O^{i_1 i_2 ... i_N}_{j_1 j_2 ... j_M} \equiv (a_1^\dagger)^{N_1} (a_2^\dagger)^{N_2} (a_3^\dagger)^{N_3} (b_1^\dagger)^{M_1} (b_2^\dagger)^{M_2} (b_3^\dagger)^{M_3} \ , \tag{38}$$

where (N_i, M_i) denote all the possible eigenvalues of the occupation number operators $(a_i^\dagger a_i, b_i^\dagger b_i)$ satisfying

$$N_1 + N_2 + N_3 = N, \quad \text{and} \quad M_1 + M_2 + M_3 = M \ . \tag{39}$$

The action of (38) on the vacuum is given by

$$O^{N_1 N_2 N_3}_{M_1 M_2 M_3} \ |\vec{0},\vec{0}> \ = \ (N_1! N_2! N_3! M_1! M_2! M_3!)^{1/2} \ |^{N_1 N_2 N_3}_{M_1 M_2 M_3}> \ . \tag{40}$$

We can now write the basis vectors of the representation (N, M) as

$$\begin{aligned} |\psi>^{i_1,i_2,...i_N}_{j_1,j_2,...,j_M} \ \equiv \ |\psi>^{N_1 N_2 N_3}_{M_1 M_2 M_3} = \Big[O^{N_1 N_2 N_3}_{M_1 M_2 M_3} + \sum_{q=1}^{Q} L_q \sum_{[\vec{\alpha}]_q} {}^{N_1}C_{\alpha_1} \ {}^{N_2}C_{\alpha_2} \ {}^{N_3}C_{\alpha_3} \\ {}^{M_1}C_{\alpha_1} \ {}^{M_2}C_{\alpha_2} \ {}^{M_3}C_{\alpha_3} \alpha_1! \alpha_2! \alpha_3! O^{N_1-\alpha_1 N_2-\alpha_2 N_3-\alpha_3}_{M_1-\alpha_1 M_2-\alpha_2 M_3-\alpha_3} \Big] |\vec{0},\vec{0}> . \end{aligned} \tag{41}$$

In this equation, $[\vec{\alpha}]_q$ denotes the sets of three non-negative integers $(\alpha_1, \alpha_2, \alpha_3)$ satisfying $\alpha_1 + \alpha_2 + \alpha_3 = q$, and $N_i - \alpha_i \geq 0$, $M_i - \alpha_i \geq 0$ for $i = 1,2,3$. The $\sum_{[\vec{\alpha}]_q}$ denotes a summation over all sets of three such integers. In the notation of Eq. (41), the tracelessness condition (37) for the $(N+1, M+1)$ representation takes the form

$$\sum_{[\vec{\gamma}]_1} |\psi>^{N_1+\gamma_1 \ N_2+\gamma_2 \ N_3+\gamma_3}_{M_1+\gamma_1 \ M_2+\gamma_2 \ M_3+\gamma_3} \ = \ 0 \ . \tag{42}$$

The definition in (41) satisfies the condition given in (42). This can be verified by using the identity

$$\begin{aligned} \sum_{[\vec{\gamma}]_1} \sum_{[\vec{\alpha}]_q} \alpha_1! \alpha_2! \alpha_3! {}^{N_1+\gamma_1}C_{\alpha_1} \ {}^{N_2+\gamma_2}C_{\alpha_2} \ {}^{N_3+\gamma_3}C_{\alpha_3} \ {}^{M_1+\gamma_1}C_{\alpha_1} \ {}^{M_2+\gamma_2}C_{\alpha_2} \ {}^{M_3+\gamma_3}C_{\alpha_3} \\ O^{N_1+\gamma_1-\alpha_1 N_2+\gamma_2-\alpha_2 N_3+\gamma_3-\alpha_3}_{M_1+\gamma_1-\alpha_1 M_2+\gamma_2-\alpha_2 M_3+\gamma_3-\alpha_3} = \Big[\Big(N+M+2-q \Big) \sum_{[\vec{\alpha}]_{q-1}} \ + \ (\vec{a}^\dagger \cdot \vec{b}^\dagger) \ \sum_{[\vec{\alpha}]_q} \Big] \\ \alpha_1! \alpha_2! \alpha_3! \ {}^{N_1}C_{\alpha_1} \ {}^{N_2}C_{\alpha_2} \ {}^{N_3}C_{\alpha_3} \ {}^{M_1}C_{\alpha_1} \ {}^{M_2}C_{\alpha_2} \ {}^{M_3}C_{\alpha_3} O^{N_1-\alpha_1 N_2-\alpha_2 N_3-\alpha_3}_{M_1-\alpha_1 M_2-\alpha_2 M_3-\alpha_3} \ . \end{aligned} \tag{43}$$

The dimension $D(N,M)$ of the representation (N,M) can be obtained as follows. For the $(N,0)$ representation, no tracelessness condition needs to be imposed, and the dimension is simply given by the number of states in Eq. (40) which satisfy $\sum_i N_i = N$ and $\sum_i M_i = 0$. This gives $D(N,0) = (N+1)(N+2)/2$. Similarly, $D(0,M) = (M+1)(M+2)/2$. Now $D(N,M)$ is given by the number of states satisfying $\sum_i N_i = N, \ \sum_i M_i = M$, which is equal to the product $D(N,0)D(0,M)$, *minus* the number of states satisfying $\sum_i N_i = N-1, \ \sum_i M_i = M-1$, which is equal to $D(N-1,0)D(0,M-1)$; the subtraction is because of the tracelessness condition. This gives

$$D(N,M) \ = \ \frac{1}{2} \ (N+1)(M+1)(N+M+2) \ . \tag{44}$$

4 $SU(3)$ Coherent States

We now observe that the states in Eq. (35) can be extracted from the following generating function,

$$|\vec{z},\vec{w}>_{(N,M)} \ \equiv \ \sqrt{N!M!} \ \exp \ (\vec{z}\cdot\vec{a}^\dagger \ + \ \vec{w}\cdot\vec{b}^\dagger) \ |\vec{0},\vec{0}> \ , \tag{45}$$

where we have to project onto the subspace of states with $\vec{a}^\dagger\cdot\vec{a} = N$ and $\vec{b}^\dagger\cdot\vec{b} = M$ to obtain the representation (N,M). More explicitly,

$$\begin{aligned} |\vec{z},\vec{w}>_{(N,M)} \quad &= \quad \frac{(\vec{z}\cdot\vec{a}^\dagger)^N}{\sqrt{N!}} \ \frac{(\vec{w}\cdot\vec{b}^\dagger)^M}{\sqrt{M!}} \ |\vec{0},\vec{0}> \\ &= \quad \sum_{N_1,N_2,N_3}{}' \ \sum_{M_1,M_2,M_3}{}' \ F_{\vec{N},\vec{M}}\,(z_1,z_2,z_3;w_1,w_2,w_3) \ \left|{}^{N_1N_2N_3}_{M_1M_2M_3}\right> \ . \end{aligned} \tag{46}$$

In (46), $\sum'$ implies that the occupation numbers (N_i,M_i) satisfy Eq. (39), and $F_{\vec{N},\vec{M}}\,(\vec{z},\vec{w})$ are given by

$$F_{\vec{N},\vec{M}}\,(\vec{z},\vec{w}) = \left(\frac{N!M!}{N_1!N_2!N_3!M_1!M_2!M_3!}\right)^{1/2} \ z_1^{N_1}z_2^{N_2}z_3^{N_3}w_1^{M_1}w_2^{M_2}w_3^{M_3} \ . \tag{47}$$

On expanding the right hand side of (46), the coefficients of $z_1^{N_1}z_2^{N_2}z_3^{N_3}w_1^{M_1}w_2^{M_2}w_3^{M_3}$ give the basis vectors of $SU(3)$ in the representation (N,M). It is important to note that the tracelessness conditions in Eq. (35) are *automatically* satisfied by the state in (46). This is because we can always replace $\left|{}^{N_1N_2N_3}_{M_1M_2M_3}\right>$ by the SU(3) basis vectors $|\psi>^{N_1N_2N_3}_{M_1M_2M_3}$ defined in (41).

It is instructive to consider a specific example here. The coherent state of the representation $(1,1)$, i.e., the adjoint representation of $SU(3)$, is given by

$$|\vec{z},\vec{w}>_{(1,1)} \ = \ \sum_{i,j=1}^{3} \ z_iw_j \ a_i^\dagger b_j^\dagger \ |\vec{0},\vec{0}> \ . \tag{48}$$

We then see that the sum of the coefficients of the three states $\left|{}^{100}_{100}\right>$, $\left|{}^{010}_{010}\right>$ and $\left|{}^{001}_{001}\right>$ is zero due to the constraint in Eq. (29). Hence there are only eight linearly independent states on

the right hand side of Eq. (48) as there should be; these eight states can be taken to be

$$
\begin{aligned}
|V_1> &= \frac{1}{\sqrt{2}} \left(|^{100}_{100}> \ - \ |^{010}_{010}>\right) , \\
|V_2> &= \frac{1}{\sqrt{6}} \left(|^{100}_{100}> \ + \ |^{010}_{010}> \ - \ 2\,|^{001}_{001}>\right) , \\
|V_3> &= |^{100}_{010}> \ , \quad |V_4> \ = \ |^{010}_{100}> \ , \\
|V_5> &= |^{100}_{001}> \ , \quad |V_6> \ = \ |^{001}_{100}> \ , \\
|V_7> &= |^{010}_{001}> \ , \quad |V_8> \ = \ |^{001}_{010}> \ .
\end{aligned} \tag{49}
$$

The states defined in Eq. (46) will be called the coherent state of the representation (N, M). Note that the equations (39), (46), (47) are analogous to the corresponding SU(2) equations (13), (12) and (14) respectively. The SU(3) coherent states (46) are normalized to unity, i.e.,

$$
{}_{(N,M)}<\vec{z},\vec{w}|\vec{z},\vec{w}>_{(N,M)} \ = \ 1 \ . \tag{50}
$$

To prove this, we use the operator identities

$$
e^A \, e^B \ = \ e^B \, e^A \, e^{[A,B]} \ , \quad \text{and} \quad e^A \, B e^{-A} \ = \ B \ + \ [A,B] \ , \tag{51}
$$

which hold if $[A, B]$ commutes with both A and B. We find that

$$
<\vec{0},\vec{0}| \ \exp\ [\vec{\bar{z}}\cdot\vec{a} + \vec{\bar{w}}\cdot\vec{b}] \ \exp\ [\vec{z}\cdot\vec{a}^\dagger + \vec{w}\cdot\vec{b}^\dagger] \ |\vec{0},\vec{0}> \ = \ \exp\ [\vec{\bar{z}}\cdot\vec{z} \ + \ \vec{\bar{w}}\cdot\vec{w}] \ . \tag{52}
$$

On comparing terms of order $(\vec{\bar{z}}\cdot\vec{z})^N(\vec{\bar{w}}\cdot\vec{w})^M$ on both sides of this equation and using the definition in (46), we obtain Eq. (50). In the same way, we can show that

$$
{}_{(N,M)}<\vec{z},\vec{w}|\vec{z}+d\vec{z},\vec{w}+d\vec{w}>_{(N,M)} \ = \ 1 \ + \ N \sum_i \bar{z}_i dz_i \ + \ M \sum_i \bar{w}_i dw_i \ , \tag{53}
$$

where $d\vec{z}$ and $d\vec{w}$ denote small deviations from $\vec{z}$ and $\vec{w}$. This equation will be used to derive the path integral formalism [4, 5] in section 5, and it would also be useful for obtaining the geometric phase for systems with $SU(3)$ symmetry [9].

We can prove that the states defined in Eq. (46) satisfy the resolution of identity, i.e,

$$
\int d\Omega \ |\vec{z},\vec{w}>_{(N,M)} \ {}_{(N,M)}<\vec{z},\vec{w}| \ = \ \frac{1}{D(N,M)} \sum_{i=1}^{D(N,M)} |V_i><V_i| \ , \tag{54}
$$

where V_i denotes a set of orthonormal basis vectors of (N, M). (See Eq. (49) for the explicit example of the representation $(1, 1)$). To verify the normalization on the right hand side of Eq. (54), it is convenient to look at a particular basis vector $|^{N00}_{0M0}>$. (This has the maximum eigenvalue $(N + M)/2$ of the operator Q^3 given in Eq. (32)). From Eq. (46), the coefficient of this vector in the coherent state is given by $z_1^N w_2^M$. Integrating the modulus squared of this using Eqs. (21 - 25), we obtain the factor of $1/D(N, M)$ in Eq. (54). This is as it should be so that taking the trace of both sides of (54) gives unity.

A second property of coherent states is that they are overcomplete. This is clear for the states in (46) since they are continuous functions of the complex variables $(\vec{z}, \vec{w})$, while the dimension of the representation (N, M) is finite.

The coherent states in (46) have a third property which is group theoretical, and is analogous to Eq. (18) for the $SU(2)$ coherent states. Namely, we can go from a particular coherent state, say, $|z_1 = 1, w_2 = 1 >_{(N,M)} = |^{N00}_{0M0} >$ to the general coherent state $|z, w >_{(N,M)}$by acting with an exponential of certain combinations of the $SU(3)$ generators Q^a. First of all, we can check that

$$|z, w >_{(N,M)} = z_1^N w_2^M \ \exp \ \left[\frac{z_2}{z_1} \ a_2^\dagger a_1 + \frac{z_3}{z_1} \ a_3^\dagger a_1 + \frac{w_1}{w_2} \ b_1^\dagger b_2 + \frac{w_3}{w_2} \ b_3^\dagger b_1\right] \ |z_1 = 1, w_2 = 1 >_{(N,M)} \ . \tag{55}$$

Then we can use Eq. (51) and the constraint (29) to rewrite this in the form [8]

$$\begin{aligned} & |z, w >_{(N,M)} \\ & = z_1^N w_2^M \ \exp \ \left[\frac{z_2}{z_1} \ (Q^1 - iQ^2) + \frac{z_3}{z_1} \ (Q^4 - iQ^5) - \frac{w_3}{w_2} \ (Q^6 + iQ^7)\right] \ |z_1 = 1, w_2 = 1 >_{(N,M)}, \end{aligned} \tag{56}$$

which is similar in structure to Eq. (18).

Another property of these coherent states which is important for their path integral applications is that the expectation value of the $SU(3)$ operators (32) in a coherent state should be given by an $SU(3)$ covariant function of $(\vec{z}, \vec{w})$ and their complex conjugates. We find that

$${}_{(N,M)} < \vec{z}, \vec{w}|Q^a|\vec{z}, \vec{w} >_{(N,M)} \ = \ N \ \bar{z}_i \lambda^a_{ij} z_j \ - \ M \ \bar{w}_i \lambda^{*a}_{ij} w_j \ . \tag{57}$$

This can be proved by using the identities in Eq. (51) to show that

$$< \vec{0}, \vec{0}| \ \exp \ [\vec{\bar{z}} \cdot \vec{a} + \vec{\bar{w}} \cdot \vec{b}] \ a_i^\dagger \ a_j \ \exp \ [\vec{z} \cdot \vec{a}^\dagger + \vec{w} \cdot \vec{b}^\dagger] \ |\vec{0}, \vec{0} > \ = \ \bar{z}_i z_j \ \exp \ [\vec{\bar{z}} \cdot \vec{z} \ + \ \vec{\bar{w}} \cdot \vec{w}] \ , \tag{58}$$

and a similar identity for the expectation value of $b_i^\dagger b_j$ in terms of $\bar{w}_i w_j$. Eq. (57) can now be obtained by comparing terms of order $\bar{z}^N z^N \bar{w}^M w^M$ on the two sides of Eq. (58).

The stationary subgroup of the coherent states defined in this section is generally $U(1) \times U(1)$, corresponding to multiplying the vectors $\vec{z}$ and $\vec{w}$ by independent phase factors. These coherent states are therefore functions of the coset space $SU(3)/U(1) \times U(1)$ [8]. However, for the completely symmetric representations $(N, 0)$ and $(0, M)$, the coherent states use only three complex numbers ($\vec{z}$ or $\vec{w}$) which define the space S^5; the stationary subgroup is then $U(1) = S^1$ which corresponds to multiplying that complex vector by a phase factor. In those cases, the coherent states are functions of the manifold S^5/S^1.

5 An Alternative Definition of $SU(3)$ Coherent States

The $SU(3)$ coherent states discussed in section 4 involve eight real parameters, and satisfy some simple group theoretic properties similar to the $SU(2)$ coherent states of section 2. It

is possible that there may be some applications of coherent states which do not require so many parameters. In this section, we will discuss an alternative kind of coherent states which only require five real parameters. We will see later that these coherent states suffer from some problems and they seem to lack some of the group theoretic properties precisely because they use fewer parameters.

We observe that the states in (35) can be extracted from the following generating function

$$|\vec{z},\vec{\bar{z}}>\equiv \exp\ (\vec{z}\cdot\vec{a}^\dagger)\ \exp\ (\vec{\bar{z}}\cdot\vec{b}^\dagger)\ \Big[\ 1+\sum_{q=1}^{Q} L_q\ \Big]\ |\vec{0},\vec{0}>\ \ , \tag{59}$$

and we have to project onto the subspace of states with $\vec{a}^\dagger\cdot\vec{a}=N$ and $\vec{b}^\dagger\cdot\vec{b}=M$ to obtain the representation (N,M). To be explicit,

$$|\vec{z},\vec{\bar{z}}>_{(N,M)}\ \ =\ \Big[\frac{(\vec{z}\cdot\vec{a}^\dagger)^N}{N!}\ \frac{(\vec{\bar{z}}\cdot\vec{b}^\dagger)^M}{M!}\ +\ \sum_{q=1}^{Q}\ L_q\ \frac{(\vec{z}\cdot\vec{a}^\dagger)^{N-q}}{(N-q)!}\ \frac{(\vec{\bar{z}}\cdot\vec{b}^\dagger)^{M-q}}{(M-q)!}\ \Big]|\vec{0},\vec{0}>\ \ . \tag{60}$$

On expanding the right hand side of (60), the coefficients of the tensors $z_{i_1}z_{i_2}...z_{i_N}\bar{z}_{j_1}\bar{z}_{j_2}...\bar{z}_{j_M}$ give the basis vectors of $SU(3)$ in the representation (N,M).

The $SU(3)$ coherent states in the representation (N,M) are defined as in Eq. (60),

$$\begin{aligned}|\vec{z},\vec{\bar{z}}>_{(N,M)}\ \ &\equiv\ \ \frac{1}{N!M!}\sum_{i_1,i_2,...}\sum_{j_1,j_2,...} z_{i_1}z_{i_2}...z_{i_N}\bar{z}_{j_1}\bar{z}_{j_2}...\bar{z}_{j_M}|\psi>^{i_1i_2...i_N}_{j_1j_2...j_M}\\ &=\ \ \sum_{N_1,N_2,N_3}\sum_{M_1,M_2,M_3}\frac{z_1^{N_1}z_2^{N_2}z_3^{N_3}\bar{z}_1^{M_1}\bar{z}_2^{M_2}\bar{z}_3^{M_3}}{N_1!N_2!N_3!M_1!M_2!M_3!}\ |\psi>^{N_1N_2N_3}_{M_1M_2M_3}\ \ .\end{aligned} \tag{61}$$

To give a specific example, the coherent state of the representation $(1,1)$ is given by

$$|\vec{z},\vec{\bar{z}}>_{(1,1)}\ \ =\ \ \sum_{i,j=1}^{3}\ z_i\bar{z}_j a_i^\dagger b_j^\dagger|\vec{0},\vec{0}>\ \ -\ \frac{1}{3}\ \sum_{i=1}^{3}\ a_i^\dagger b_i^\dagger|\vec{0},\vec{0}>\ \ . \tag{62}$$

We will now prove that the states defined in (61) satisfy the resolution of identity,

$$\int\ d\Omega_{S^5}\ |\vec{z},\vec{\bar{z}}>_{(N,M)}\ {}_{(N,M)}<\vec{z},\vec{\bar{z}}|\ =\ 1\ . \tag{63}$$

To prove this, we use the the definition (41) and the integration measure for $\vec{z}$ given in (22). We find that

$$\begin{aligned}&\int d\Omega_{S^5}\ |z,\bar{z}><z,\bar{z}|\\ &=\ \mathcal{C}\sum_{N_i,M_i}\Big(\sum_{\delta_i}\big(\prod_{i=1}^{3}\frac{(N_i+M_i+\delta_i)!}{(N_i+\delta_i)!(M_i+\delta_i)!}\big)\ |\psi>^{N_1+\delta_1N_2+\delta_2N_3+\delta_3}_{M_1+\delta_1M_2+\delta_2M_3+\delta_3}\Big)\ {}^{N_1N_2N_3}_{M_1M_2M_3}<\psi|\ ,\end{aligned} \tag{64}$$

where the δ_i are integers satisfying

$$\sum_{i=1}^{3}\ \delta_i\ =\ 0\ , \tag{65}$$

and the constant $\mathcal{C}$ is determined below. We now use the following property

$$\sum_{\delta_i} (\prod_{i=1}^{3} \frac{(N_i+M_i+\delta_i)!}{(N_i+\delta_i)!(M_i+\delta_i)!}) \; |\psi >^{N_1+\delta_1 N_2+\delta_2 N_3+\delta_3}_{M_1+\delta_1 M_2+\delta_2 M_3+\delta_3} = |\psi >^{N_1 N_2 N_3}_{M_1 M_2 M_3} \;\; , \tag{66}$$

which is a consequence of Eq. (37) for the basis vectors of a representation of $SU(3)$. Thus Eq. (64) can be simplified to

$$\int d\Omega_{S^5} \; |z,\bar{z}><z,\bar{z}| \;=\; \mathcal{C} \sum_{N_i,M_i} |\psi >^{N_1 N_2 N_3}_{M_1 M_2 M_3} \;\; {}^{N_1 N_2 N_3}_{M_1 M_2 M_3} < \psi| \; . \tag{67}$$

The normalization constant $\mathcal{C}$ in Eq. (67) can be fixed by looking at one particular basis vector of the representation (N,M), say,

$$|\psi >^{N00}_{0M0} \;\; . \tag{68}$$

From Eq. (61), the coefficient of this vector in the coherent state $|\vec{z},\vec{\bar{z}}>$ is $z_1^N \bar{z}_2^M/(N!M!)$. Integrating this as in (22), we find that

$$\mathcal{C} \;=\; \frac{2}{N!M!(N+M+2)!} \; . \tag{69}$$

Finally, let us consider the analog of the property given in Eq. (57) for the (z,w) coherent states. We can prove that

$${}_{(N,M)}<\vec{z},\vec{\bar{z}}|Q^a|\vec{z},\vec{\bar{z}}>_{(N,M)} \;\;=\;\; (N-M) \; \bar{z}_i \lambda^a_{ij} z_j \; . \tag{70}$$

To prove this, we use the identities in (51) to show that

$$<\vec{0},\vec{0}| \;\exp\; [\vec{\bar{z}}\cdot\vec{a}+\vec{z}\cdot\vec{b}] \; a_i^\dagger \; a_j \;\exp\; [\vec{z}\cdot\vec{a}^\dagger+\vec{\bar{z}}\cdot\vec{b}^\dagger] \; |\vec{0},\vec{0}> \;\;=\;\; \bar{z}_i z_j \;\exp\; [2\vec{\bar{z}}\cdot\vec{z}] \; . \tag{71}$$

On expanding this equation and comparing terms which are of order N in both z_i and $\bar{z}_i$, we find that the expectation value of Q^a in the representation $(N,0)$ satisfies Eq. (70). In a similar way, we can prove Eq. (70) in the representation $(0,M)$. Finally, we can generalize the proof to the representation (N,M) by using Eq. (33); since Q^a commutes with $\vec{a}\cdot\vec{b}$ and $\vec{a}^\dagger\cdot\vec{b}^\dagger$, it also commutes with the operators L_q which are require to enforce tracelessness in Eq. (35).

Note that (70) vanishes for the self-conjugate representations in which $N=M$. There is a similar problem for the differential change in overlap analogous to Eq. (53). We find that the coherent states defined in this section satisfy

$$\frac{<\vec{z},\vec{\bar{z}}|\vec{z}+d\vec{z},\vec{\bar{z}}+d\vec{\bar{z}}>}{<\vec{z},\vec{\bar{z}}|\vec{z},\vec{\bar{z}}>} \;=\; 1 \;+\; N \sum_i \; \bar{z}_i dz_i \;+\; M \sum_i d\bar{z}_i z_i \tag{72}$$

in the representation (N,M). The left hand side of this equation is equal to 1 if $N=M$ due to the constraint $\sum_i \bar{z}_i z_i = 1$. These two problems imply that the $(z,\bar{z})$ coherent states are unlikely to be useful for path integral applications in the representations with $N=M$.

For the $(z, \bar{z})$ coherent states, we have not yet found the construction of the group theoretical property analogous to (56) in the general representation (N, M). This would be an interesting topic for future studies.

The stationary subgroup of the coherent states defined in this section is $U(1) = S^1$, corresponding to multiplying $\vec{z}$ by a phase factor. These coherent states are therefore functions of the manifold S^5/S^1.

6 Path Integral Formalism

We will now use the (z, w) coherent states presented in section 4 to derive the path integral for a problem which has $SU(3)$ variables in some representation (N, M). (For convenience, we will drop the subscript (N, M) on the coherent states in this section). We begin by discussing a problem involving the Hamiltonian of a single site with a $SU(3)$ variable. For any Hamiltonian which is a function of the $SU(3)$ operators Q^a, we define its coherent state expectation value to be

$$E(z, \bar{z}, w, \bar{w}) \;\equiv\; < z, w|\hat{H}|z, w > \;\; . \tag{73}$$

If the Hamiltonian is linear in the $SU(3)$ operators, i.e.,

$$\hat{H} \;=\; \sum_{a=1}^{8} c_a Q^a \;, \tag{74}$$

then Eq. (73) can be found found using Eq. (57). But if the Hamiltonian is not linear in the $SU(3)$ operators, then Eq. (73) has to be evaluated separately.

Let us now consider the propagator in imaginary time

$$G(z^{(F)}, w^{(F)}, z^{(I)}, w^{(I)}; T) \;=\; < z^{(F)}, w^{(F)}| \exp(-T\hat{H})|z^{(I)}, w^{(I)} > \;\; , \tag{75}$$

where the superscripts I and F denote initial and final states respectively, and we are suppressing the subscripts i $(= 1, 2, 3)$ on z and w for the moment. We write the exponential in (75) as a product of $\mathcal{N}$ terms, and use the resolution of identity in (54) to insert a complete set of states between each pair of terms. A typical term looks like

$$< z^{(n+1)}, w^{(n+1)}| \exp(-\epsilon\hat{H})|z^{(n)}, w^{(n)} > \;\; , \tag{76}$$

where $\epsilon = T/\mathcal{N}$. We are eventually interested in taking the limit $\mathcal{N} \to \infty$ holding T fixed. In that case, we may assume that $(z^{(n+1)}, w^{(n+1)})$ is close to $(z^{(n)}, w^{(n)})$ in (76), so that $dz_i^{(n)} = z_i^{(n+1)} - z_i^{(n)}$ and $dw_i^{(n)} = w_i^{(n+1)} - w_i^{(n)}$ are small. Using Eqs. (53) and (73), we can write (76) as

$$\begin{aligned} &< z^{(n+1)}, w^{(n+1)}| \exp(-\epsilon\hat{H})|z^{(n)}, w^{(n)} > \\ &= \;\exp[N \sum_i \bar{z}_i^{(n)} dz_i^{(n)} \;+\; M \sum_i \bar{w}_i^{(n)} dw_i^{(n)} \;-\; \epsilon E(z^{(n)}, \bar{z}^{(n)}, w^{(n)}, \bar{w}^{(n)})] \end{aligned} \tag{77}$$

to first order in ϵ, $dz_i^{(n)}$ and $dw_i^{(n)}$. In the limit $\epsilon = d\tau \to 0$, we can write the propagator in (75) in the path integral form

$$
\begin{aligned}
G(z^{(F)}, w^{(F)}, z^{(I)}, w^{(I)}; T) &= \int \mathcal{D}\Omega_{SU(3)}(\tau) \ \exp(-S[z,w]) \ , \\
\text{where} \quad S[z,w] &= \int_0^T d\tau \ [- \ N \sum_i \bar{z}_i \frac{dz_i}{d\tau} \ - \ M \sum_i \bar{w}_i \frac{dw_i}{d\tau} \ + \ E(z, \bar{z}, w, \bar{w})] \ , \\
\text{and} \quad \mathcal{D}\Omega_{SU(3)}(\tau) &\equiv \prod_n \ d\Omega_{SU(3)}(n) \ ,
\end{aligned} \tag{78}
$$

and (z, w) are functions of τ which satisfy the boundary conditions $(z(0), w(0)) = (z^{(I)}, w^{(I)})$ and $(z(T), w(T)) = (z^{(F)}, w^{(F)})$. Note that we have written the functional integral measure in (78) in terms of the measure given in Eq. (25). Alternatively, we can write the functional integral measure in terms of $\mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}w\mathcal{D}\bar{w}$ if we introduces appropriate Lagrange multiplier fields in the action S to enforce the constraints in Eqs. (28 - 29) at each time τ.

We can now generalize the above construction to a problem involving several sites which are labelled by a parameter x, provided that the Hamiltonian is linear in the $SU(3)$ variables at *each* site. We introduce a coherent state at each site, and write the energy functional as

$$
\begin{aligned}
E[z, \bar{z}, w, \bar{w}] &= \ < z, w|\hat{H}|z, w > \ , \\
\text{where} \quad |z, w> &\equiv \prod_x \ |z(x), w(x) > \ .
\end{aligned} \tag{79}
$$

Then we can show that

$$
\begin{aligned}
&< z^{(F)}(x), w^{(F)}(x)| \exp(-T\hat{H})|z^{(I)}(x), w^{(I)}(x) > \ = \int \mathcal{D}\Omega_{SU(3)}(x, \tau) \ \exp(-S[z,w]) \ , \\
&S[z,w] \ = \int_0^T d\tau \ \left[\ - \sum_x \ \{ \ N \sum_i \bar{z}_i(x) \frac{dz_i(x)}{d\tau} \ - \ M \sum_i \bar{w}_i(x) \frac{dw_i(x)}{d\tau} \ \} \ + \ E[z, \bar{z}, w, \bar{w}] \right], \\
&\mathcal{D}\Omega_{SU(3)}(x, \tau) \equiv \prod_{x,n} \ d\Omega_{SU(3)}(x, n) \ .
\end{aligned} \tag{80}
$$

Note that the first two terms in the actions S given in Eqs. (78) and (80) are purely imaginary due to the constraints in (28). To show this explicitly, we can rewrite those terms as

$$
\begin{aligned}
\sum_i \ \bar{z}_i dz_i &= \frac{1}{2} \sum_i \ (\ \bar{z}_i dz_i \ - \ d\bar{z}_i z_i \ ) \ , \\
\text{and} \quad \sum_i \ \bar{w}_i dw_i &= \frac{1}{2} \sum_i \ (\ \bar{w}_i dw_i \ - \ d\bar{w}_i w_i \ ) \ .
\end{aligned} \tag{81}
$$

As an example of a problem to which this formalism can be applied, we can consider the $SU(3)$ invariant Hamiltonian

$$
\hat{H} \ = \ \sum_{x,y} \ J_{x,y} \ \sum_a \ Q^a(x) Q^a(y) \ . \tag{82}
$$

This is called the $SU(3)$ Heisenberg model. It has been discussed extensively in the literature for the completely symmetric representations $(N, 0)$ [4]; for those representations, we can use the simpler measure $d\Omega_{S^5}$ given in Eq. (22) instead of $d\Omega_{SU(3)}$. Our construction of coherent states now allows a study of the Heisenberg model in any representation (N, M).

7 Summary and Discussion

In this paper we have exploited the representation of the $SU(3)$ Lie algebra in terms of six harmonic oscillator creation and annihilation operators to generate all the representations of $SU(3)$. This harmonic oscillator form of the algebra enables us to define the $SU(3)$ coherent states in terms of two triplets of complex numbers. In this sense the $SU(2)$ (12) and $SU(3)$ definitions (45) are analogous to that of the Heisenberg-Weyl coherent states (4). The $SU(3)$ coherent states are characterized by two triplets of complex numbers with 4 real constraints. This explicit construction in terms of complex numbers can be used to derive the geometrical phase of $SU(3)$. Further, the path integral formalism discussed in the previous section can be used to obtain the field theory for the $SU(3)$ Heisenberg model and study its topological aspects as in the $SU(2)$ case [14]. Work in this direction is in progress and will be reported elsewhere.

For any group G, we can use a certain number of harmonic oscillator operators to construct the group operators as in Eqs. (32) and (33). If we can find the appropriate set of complex numbers which transform according to that group and satisfy the necessary constraints, we can use our method to provide an explicit complex number parameterization of the corresponding coherent states.

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